Table 1 Incipient separation data

M_1	$\Theta_i^{(o)}$	$\beta^{(o)}$	M_n	P_i/P_1	$M_1\Theta_i$
1.96[a]	8.5	38.5	1.22	1.58	0.290
2.50[b]	7.5	29.5	1.23	1.60	0.325
2.94[a]	5.5	24.0	1.20	1.50	0.282
3.44[b]	5.0	20.5	1.20	1.50	0.300

[a] Data from Ref. 1.
[b] Data from Ref. 2.

experimental points represent a conservative assumption of $\pm 0.5°$ accuracy of the data.

Substituting $\sin\beta = M_n/M_1$ for the shock angle into the oblique shock wave relation, and making the small angle approximation corresponding to $M_1 \gg M_n$ or 1, Eq. (4) is expressed, to a first order, by

$$M_1\Theta = [M_n^2 - 1]\bigg/\left[\left(\frac{\gamma+1}{2}\right)M_n\right] \tag{8}$$

For the incipient case of Eq. (7), one obtains $M_n = 1.20$ corresponding to which $P_i/P_1 = 1.50$. The error in M_n for the incipient case as calculated from Eq. (8) compared with the exact value given by Eq. (4) is less than 3% for $M \geqq 2$.

It is of interest to note that incipient turbulent boundary-layer separation for normal shock interaction occurs at a Mach number of about 1.3 according to Fage and Sargent,[3] for which the pressure rise is about 1.80. This suggests that a skewed shock gives rise to somewhat earlier separation than a normal shock. A conjecture for this behavior is that the surface streamlines are forced into a complete reversal in direction in the two-dimensional case, whereas with a skewed shock they are only pressed into a small change in direction in the process of coalescing along a separation line.

Summary and Discussion

In summary, a simple correlation is obtained for incipient turbulent boundary-layer separation due to the impingment of a skewed shock wave, as follows:

$$M_1\Theta_i = 0.30 \qquad P_i/P_1 = 1.50$$

Thus incipient separation is associated with an approximately constant pressure rise independent of Mach number and the incipient separation deflection angle varies inversely with Mach number at least over the range of available experimental data. The validity of this correlation needs checking by experimental data at Mach numbers above 3.5.

It is interesting to note that this behavior is contrary to that for two-dimensional turbulent boundary-layer separation for which the incipient deflection angle increases with increasing Mach number, and the corresponding pressure rise increases at an even steeper rate.

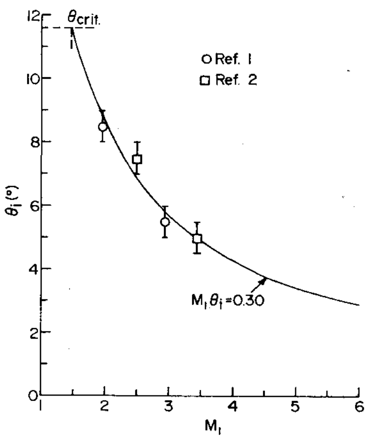

Fig. 2 Correlation for flow deflection angle for incipient separation.

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Near-Field Trajectory of Turbulent Jets Discharged at Various Inclinations into a Uniform Crossflow

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Introduction

THE deflected flow of turbulent jets in cross streams has been extensively investigated by environmental engineers in connection with the discharge of effluents from chimney stacks into the atmosphere or of liquid pollutants into natural bodies of water. Their interest inclines toward the study of dispersion of effluents or pollutants at far field, away from the discharge point. Recent interest in such interacting flows also has arisen from the design of V/STOL aircrafts. During the transition from hovering to forward flight, such aircraft produce an analogous deflected-jet flow, which, in turn, can have serious consequences on the dynamic behavior of the aircraft; the interest here is the near-field flow of deflected jets.

A systematic experiment was conducted by Platten and Keffer[1] of axisymmetric turbulent jets discharging at various angles of inclination into a uniform cross stream having various ratios between jet and stream velocities. On the basis of their results, it is shown herein that jet trajectories in the near field follow a universal law by which the variations in both the angle of inclination and the velocity ratio can be accommodated.

Turbulent Jets Ejected Normally into Cross Streams

The flow of a turbulent jet discharging normally into a cross stream was considered by Pratte and Baines[3] to exhibit three sequential patterns of development: 1) a potential core which exists before the turbulent shear region developed along the jet boundary reaches the centerline of the jet; 2) a zone of maximum deflection in the near field where rapid entrainment, similar to that for a free jet, occurs to cause the jet to bend over; 3) a vortex zone in the far field where the flow is governed mainly by the motion of a turbulent vortex pair formed following the bending over.

As discussed by Pratte and Baines,[3] the centerline jet trajectory, especially in the near field, can be obtained in simple form from a dimensional argument on the basis of flow similarity

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Index category: Jets, Wakes, and Viscid-Inviscid Flow Interactions.

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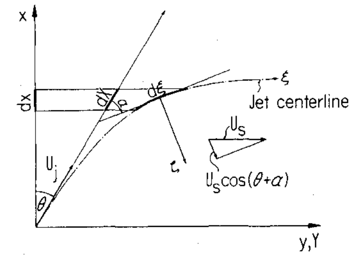

Fig. 1 Definition sketch and coordinate systems.

$$x/D = f(y/D, R), \qquad R = U_j/U_s \tag{1}$$

The notation is defined in Fig. 1, where x is the penetration of the jet with an initial diameter D, y is the downstream distance from the jet discharge point, and R is the ratio between the initial jet velocity U_j and the uniform stream velocity U_s. The viscous effect is neglected, either because a very high Reynolds number exists in the near field or a very large jet diameter, in comparison with the size of energy-dissipation eddies, exists in the far field. It is noted that the influence of environmental conditions on the flow in the far field is not considered here.

Adopting a curvilinear coordinate system, as shown in Fig. 1, with ξ-axis along, and ζ-axis normal to, the curved path of the jet, the momentum balance in the ζ-direction can be written as

$$\rho(U_e C d\xi)(U_s \cos\alpha) = \int_A \rho(udA)(ud\alpha) \tag{2}$$

where ρ is the density of the fluid, U_e is the entrainment velocity normal to the circumference C of the jet with a cross-sectional area A, α is the angle between the fixed and the curvilinear coordinate directions, u is the local jet velocity along the curved path. If we assume that the crossflow provides only a convection velocity for the embedded jet and does not affect the entrainment, then the following variations[4] can be expected to apply for a weak crossflow

$$C/D \sim \xi/D, \quad U_e/U_j \sim D/\xi, \quad udA/U_j D^2 \sim \xi/D, \quad u/U_j \sim D/\xi \tag{3}$$

Substituting Eq. (3) into Eq. (2), Pratte and Baines[3] obtained the following approximation:

$$d\alpha/\cos\alpha \sim d\xi/DR \quad \text{or} \quad d\alpha \sim dx/DR \tag{4}$$

which can be integrated and combined to result

$$x/DR = f_1(\xi/DR) \quad \text{or} \quad x/DR = f_2(y/DR) \tag{5}$$

It was also demonstrated experimentally by Pratte and Baines that the centerline trajectories of jets, ejected normally into cross streams and at various jet-to-stream velocity ratios, collapse to a common profile, represented by the aforementioned functional form.

A kink is exhibited in the trend of Pratte and Baines' data at $y/DR \simeq 2.5$ with more rapid variation of x/DR with y/DR occurring in the near field than in the far field. The changeover may be interpreted to indicate the transition of the flow from the zone of maximum deflection to the final vortex zone. However, such a trend was ignored by Pratte and Baines and a single straightline was used to represent the entire profile. Because the far-field data extended as far as $y/DR \simeq 300$ while the near-field data stayed within $y/DR < 2.5$, the power law derived by them, therefore, puts more weight on the far-field data. Their results in the near field are replotted as an insert in Fig. 3. A new line, shown in the insert, is fitted on the basis of the least squares principle and corresponds to the following relation

$$x/DR = 2.0(y/DR)^{0.38} \tag{6}$$

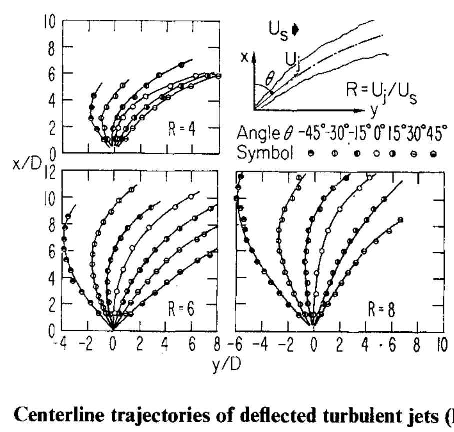

Fig. 2 Centerline trajectories of deflected turbulent jets (Ref. 1).

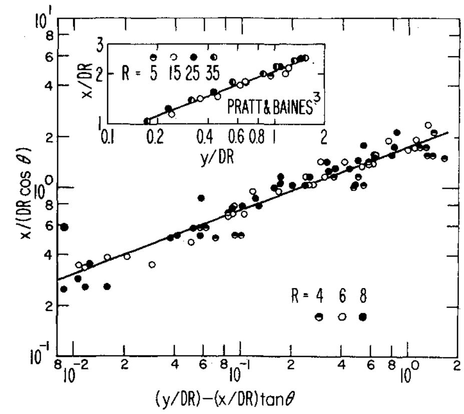

Fig. 3 Universal trajectory of deflected turbulent jets.

Turbulent Jets Ejected at Various Inclinations into Cross Streams

Recently, experiments on deflected turbulent jets, discharging at various angles of inclination into crossflowing streams, were conducted by Platten and Keffer.[1] Their results, obtained at various ratios between jet and stream velocities, are replotted in Fig. 2, wherein the solid lines are the curves fitted by the original authors. The jet profiles appear to extend well beyond the potential-core region and to encompass a major portion of the zone of maximum deflection.

It is suggested that the earlier treatment[3] of the trajectories of jets discharging normally into a cross stream can be extended to jets discharging at various inclinations into a cross stream by means of the following normalization as shown in Fig. 1: setting the X-axis along the initial-velocity vector of the jet and the Y-axis along the velocity vector of the cross stream, or

$$X = x/\cos\theta, \qquad Y = y - x\tan\theta \tag{7}$$

For an inclined jet, the equation for momentum balance becomes

$$\rho(U_e C d\xi)\cdot U_s \cos(\theta+\alpha) = \int_A \rho(udA)(ud\alpha) \tag{8}$$

Following the same arguments as in the previous section, we obtain

$$d\alpha \sim d\xi\cos(\theta+\alpha)/DR \quad \text{and} \quad dX = d\xi\cos(\theta+\alpha)/\cos\theta \tag{9}$$

For limiting cases with small θ, $\cos\theta$ approaching unity, we have

$$d\alpha \sim dX/DR \tag{10}$$

Such a limitation may be less restrictive than it appears, as $\cos\theta$ has a value of 0.9 for an angle as great as 25°. Accepting the foregoing approximation and following the rest of the procedure in the previous section, a universal trajectory can be expressed for deflected turbulent jets discharging at various angles with respect to the crossflow and having various ratios between jet and stream velocities as

$$X/DR = f_3(Y/DR) \tag{11}$$

The experimental results of Platten and Keffer are plotted in Fig. 3 using the proposed coordinates. The data points are obtained from the faired curves shown in Fig. 2 at $x/d = 2, 4,$

6, 8, 10, and 12. Despite having large range of variation of θ, the experimental results are seen to be very well correlated by the suggested parameters. The straightline, fitted on the basis of the least-square principle and shown in Fig. 3, corresponds to a power-law trajectory

$$X/DR = 1.75(Y/DR)^{0.38} \tag{12}$$

The exponent found here in Eq. (12) for inclined jets is the same as that found earlier in Eq. (6) for normal jets—representing a special case of inclined jets with $X = x$ and $Y = y$. The discrepancy between the coefficients is apparently due to the use of different devices to produce the jets.[1] A uniform initial velocity across the jet was assured by Pratte and Baines by using an orifice, whereas a pipe-flow (parabolic) velocity distribution is introduced by Platten and Keffer, who used a long tube for their means of jet injection. Such differences should not affect the exponent, however, although they can be expected to influence the coefficient of the power-law profile.

Conclusions

It has been shown here that use of a skewed coordinate system, aligning the coordinate axes with the jet and the stream velocity vectors, can greatly simplify the description of the trajectories of turbulent jets discharging at various angles of inclination into a uniform cross stream and having various ratios between the initial jet velocity and the stream velocity. By aid of this skewed coordinate system and on the basis of dimensional and similarity considerations, the centerline trajectories of such jets appear to be very well correlated by a simple power law. The centerline trajectory and induced pressure field of the deflected jets also can be determined by integrating numerically the momentum equation from the given initial condition at the jet exit and with the entrainment coefficient and the crossflow drag coefficient chosen from the test data. Such calculations certainly are important for determining the pressure field induced by the deflected jet, but the present formula should provide a convenient and sufficient estimation of the centerline trajectory of the deflected jets.

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Unsteady Boundary-Layer Flow of Power Law Fluids

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Nomenclature

A, B = dimensional constants, Eqs. (3) and (4)
f = nondimensional velocity
k = coefficient in constitutive equation for power law fluids
m = constant, Eq. (4)
n = power law fluid index
t = time
u = velocity in boundary layer
U = velocity just outside boundary layer
y = distance normal to surface
α = positive constant, Eq. (3)
ν = kinematic viscosity of Newtonian fluids
η = similarity variable
ρ = density of fluid

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THE purpose of this Note is to show that two-dimensional flows past surfaces immersed in non-Newtonian power law fluids, which start moving impulsively with velocities proportional to t^α, admit of similarity solutions. The previous analytical studies that have come to the notice of the author include the works of Bird,[1] Wells,[2] Rott[3] and Chen and Wollersheim.[4] In particular, Chen and Wollersheim studied the case $\alpha = 0$. Roy[5] performed a perturbation analysis assuming the fluid to be slightly non-Newtonian and showed that the results thus obtained compared very favorably with those predicted by Chen and Wollersheim within the range $0.5 \leqq n \leqq 1.5$.

Mathematically speaking, we have to solve the boundary-layer equation

$$\partial u/\partial t = \partial U/\partial t + (k/\rho)(\partial/\partial y)(\partial u/\partial y)^n \tag{1}$$

subject to the boundary conditions

$$\begin{aligned} u = 0, \quad & y = 0, \quad t > 0 \\ u \to U \quad \text{as} \quad & y \to \infty, \quad t > 0 \\ \text{and at} \quad & t = 0_+, \quad y > 0 \end{aligned} \tag{2}$$

Assume

$$U = At^\alpha \tag{3}$$

where A and α are constants.

These equations have similarity solutions given by

$$\left.\begin{aligned} \eta = y/Bt^m, \quad & u = Uf(\eta) \\ m = (1/n+1)&\{(n-1)\alpha+1\} \\ B = \{2n(n+1)KA^{n-1}/\rho&[(n-1)\alpha+1]\}^{1/(n+1)} \end{aligned}\right\} \tag{4}$$

where f satisfies the equation

$$\left.\begin{aligned} (m/n)(d/d\eta)(f')^n + 2m\eta f' + 2\alpha(1-f) = 0 \\ f(0) = 0, \quad f(\infty) = 1 \end{aligned}\right\} \tag{5}$$

It can be easily seen that for $\alpha = 0$ we attain the equations of Chen and Wollersheim. Also for $n = 1$ the similarity variable and similarity equations become

$$\left.\begin{aligned} \eta = y/2(\nu t)^{1/2} \\ \text{and} \\ f'' + 2\eta f' + 4\alpha(1-f) = 0 \\ f(0) = 0, \quad f(\infty) = 1 \end{aligned}\right\} \tag{6}$$

Equations (6) have been obtained and solved by Roy[6] for different values of α.

The numerical solutions of Eq. (5) are presented in Table 1 for $\alpha = \frac{1}{2}$ and 1. It is observed that the values of $f'(0)$ at first increase with n up to $n = 1$ and then decrease. Similar trends about $n = 1$ have been widely exhibited by Shah's calculations for steady flows of power law fluids.[7]

Table 1 Values of $f'(0)$ for different values of α

n	$\alpha = 1/2$	$\alpha = 1$
0.25	1.6452	1.9395
0.50	1.7142	2.1085
0.75	1.7553	2.2096
1.00	1.7725	2.2568
1.25	1.7710	2.2553
1.50	1.7579	2.2322
1.75	1.7378	2.2120
2.00	1.7133	2.0081